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SINGULARITY OF ORBITAL MEASURES IN SU(n)*

BY

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ABSTRACT

We show that the minimal k such that $\mu^k \in L^1(\mathrm{SU}(n))$ for all central, continuous measures μ on $\mathrm{SU}(n)$ is $k = n$. We do this by exhibiting an element $g \in \mathrm{SU}(n)$ for which the $(n-1)$-fold product of its conjugacy class has zero Haar measure. This ensures that if μ_g is the corresponding orbital measure supported on the conjugacy class, then μ_g^{n-1} is singular to L^1.

1. Introduction

It is well known that there are many continuous, singular measures on the circle (or any compact abelian group) all of whose convolution powers remain singular to L^1. In contrast, Ragozin in [6] proved the striking fact that if G is a compact, connected, simple Lie group and μ is any central, continuous measure on G, then $\mu^{\dim G} \in L^1(G)$.

Ragozin's result was first improved in [2] where it was shown that if $k > \dim G/2$ and μ is any central, continuous measure, then $\mu^k \in L^1(G)$, and that $\mu^k \in L^2(G)$ for many continuous central measures. This result was subsequently improved for the classical Lie groups in [3] where estimates were made on the

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size of characters (in terms of their degrees) which enabled one to determine the minimal k such that $\mu^k \in L^2(G)$ for all continuous, orbital measures μ. The precise choice of k depends upon the Lie group type, but is roughly the rank of G.

In particular, this improvement implies that when the group is $\mathrm{SU}(n)$, then $\mu^n \in L^1$ for all central, continuous measures μ. In [1] it had previously been conjectured that $\mu^{n-1} \in L^1(\mathrm{SU}(n))$. Here we will exhibit a central, continuous measure such that μ^{n-1} is purely singular, thereby completing the proof that $\mu^k \in L^1$ for all central, continuous measures μ on $\mathrm{SU}(n)$ if and only if $k \geq n$.

Our example is an orbital measure. The **orbital measure**, μ_g, supported on the conjugacy class $C(g)$ containing g, is defined by

$$\int_G f d\mu_g = \int_G f(tgt^{-1})dm_G(t) \quad \text{for any continuous function } f.$$

Orbital measures are continuous if and only if g is not in the centre of the group, and are always singular to Haar measure being supported on a submanifold of lower dimension. Indeed, μ_g^k is (trivially) singular provided $k \dim C(g) < \dim G$.

In [3] it was shown that $\mu_g^n \in L^2(\mathrm{SU}(n))$ for all g not in the centre of $\mathrm{SU}(n)$. Since μ_g^n is supported on $C(g)^n$, it follows that the Haar measure of $C(g)^n$ is positive for all such g. In this paper we will prove that the measure of $C(g)^{n-1}$ is zero when g is the diagonal matrix $(e^i, \ldots, e^i, e^{-(n-1)i})$ in $\mathrm{SU}(n)$. Because μ_g^{n-1} is supported on $C(g)^{n-1}$ this demonstrates that μ_g is a central, continuous measure whose $(n-1)$-st convolution power is singular.

2. Basic properties of tangent spaces

2.1. Outline of the proof. We begin by introducing notation and giving an outline of the strategy of the proof.

Let $G = \mathrm{SU}(n)$, $n \geq 2$, and let $\underline{\underline{g}}$ denote its Lie algebra,

$$\underline{\underline{g}} \equiv \{n \times n \text{ matrices } z : z = -z^*, \ \operatorname{tr} z = 0\}.$$

We view $\underline{\underline{g}}$ as a real algebra of dimension $n^2 - 1$. For $z = (z_{jk}) \in \underline{\underline{g}}$ we write $z_{jk} = x_{jk} + iy_{jk}$ with x_{jk}, y_{jk} real. Note that $x_{jj} = 0$ and $z_{kj} = -x_{jk} + iy_{jk}$. Initially we focus on the diagonal matrix $H = (i, \ldots, i, -(n-1)i)$ in $\underline{\underline{g}}$ and the adjoint orbit of H in $\underline{\underline{g}}$,

$$O = \{\mathrm{Ad(g)H} : \mathrm{g} \in \mathrm{G}\} = \{gHg^{-1} : g \in G\}.$$

The matrix H is relevant because its exponential is the matrix in SU(n) mentioned in the introduction.

Our approach is geometric, and involves a study of the tangent spaces to the adjoint orbit. First, properties of the orbit O are used to find small spanning sets for the tangent spaces to O at typical points of O. Next, the spanning sets are used to prove that the dimension of the sum of any $n-1$ of these 'typical' tangent spaces is less than the dimension of $\underline{\underline{g}}$. This is basically done by finding sufficiently many linearly independent vectors in the kernel of a suitable matrix. A denseness and continuity argument allows us to extend this result to the sum of any $n-1$ tangent spaces to the orbit. Because these sums of tangent spaces are the images of the differential of the addition map from O^{n-1} to the $(n-1)$-fold sum of O, Sard's theorem can be invoked to argue that the Haar measure of $(n-1)O$, and hence $\exp((n-1)O)$, is zero. Finally, we consider $g \equiv \exp H$ in SU(n), and its conjugacy class C. Since it is known that $C^{n-1} \subseteq \exp(n-1)O$, this set must also have Haar measure zero, completing the proof of our key result.

2.2. Properties of the adjoint orbit.

PROPOSITION 2.1: *The dimension of the orbit O is $2(n-1)$.*

Proof: A general fact about Lie algebras is that if Φ^+ is the set of positive roots of $\underline{\underline{g}}$ and $\Phi^+(H)$ is the set of positive roots annihilating H, then the dimension of the adjoint orbit of H is $2(|\Phi^+| - |\Phi^+(H)|)$ ([5], ch. 6: 4.8). In our case we may take

$$\Phi^+ = \{e_i - e_j : 1 \leq i < j \leq n\}$$

where $\{e_i\}_{i=1}^n$ denotes the standard basis vectors in $\mathbb{R}^n$. Then

$$\Phi^+(H) = \{e_i - e_j : 1 \leq i < j \leq n-1\}$$

which has cardinality $\binom{n-1}{2}$. ∎

Next, we list some elementary properties of elements of the orbit.

PROPOSITION 2.2: *Suppose $z \in O$. Then*

(i) $x_{kn}^2 + y_{kn}^2 = (1 - y_{kk})(1 - y_{nn})$ *for* $1 \leq k \leq n-1$,

(ii) $(1 - y_{nn})x_{jk} = -x_{jn}y_{kn} + y_{jn}x_{kn}$ *for* $1 \leq j < k \leq n-1$,

(iii) $-(1 - y_{nn})y_{jk} = x_{jn}x_{kn} + y_{jn}y_{kn}$ *for* $1 \leq j < k \leq n-1$.

Proof: Let $z = (z_{jk}) = gHg^{-1}$ for $g \in \mathrm{SU}(n)$ and assume $g = (g_{jk})$ with $g_{jk} = \alpha_{jk} + i\beta_{jk}$, α_{jk}, β_{jk} real. One can easily calculate that

$$z_{jk} = i\sum_{l=1}^{n-1} g_{jl}\overline{g_{kl}} - (n-1)ig_{jn}\overline{g_{kn}}.$$

Using the properties $\sum_{l=1}^{n}|g_{kl}|^2 = 1$ and $\sum_{l=1}^{n} g_{kl}\overline{g_{jl}} = 0$ if $j \neq k$ one can verify that the diagonal entries of z satisfy $y_{kk} = 1 - n|g_{kn}|^2$, and the non-diagonal entries are given by

$$x_{jk} = n(\alpha_{kn}\beta_{jn} - \alpha_{jn}\beta_{kn}) \quad \text{and} \quad y_{jk} = -n(\alpha_{kn}\alpha_{jn} + \beta_{jn}\beta_{kn})$$

for $j \neq k$. It is a straightforward exercise to see that the claims follow easily from these facts. ∎

We identify $z \in \underline{\underline{g}}$ as the row vector (also denoted z) in $\mathbb{R}^{n^2-1}$ with the identification

$$(2.1) \qquad z = [y_{11}, \ldots, y_{n-1,n-1}, x_{12}, y_{12}, \ldots, x_{1,n-1}, y_{1,n-1}, x_{23}, \\ \ldots, y_{2,n-1}, \ldots, x_{1,n}, \ldots, y_{n-1,n}].$$

If $y_{nn} \neq 1$ we call z **generic**.

2.3. Tangent spaces of the orbit. As mentioned earlier, our approach to the problem involves a study of tangent spaces: For $z \in O$, the tangent space to O at z will be denoted T_z. This is known ([5], ch. 6: 4; see also [9], 2.9) to equal

$$T_z = \left\{[X, z] : X \in \underline{\underline{g}}\right\}.$$

Let e_{jk} denote the $n \times n$ matrix with 1 in entry j, k and 0 else. For $1 \leq j \leq n-1$, define X_j and $Y_j \in \underline{\underline{g}}$ by $X_j = e_{jn} - e_{nj}$ and $Y_j = i(e_{jn} + e_{nj})$. We will think of $[X_j, z]$ and $[Y_j, z]$ as row vectors in $\mathbb{R}^{n^2-1}$ with the ordering outlined above, and construct a $2(n-1) \times (n^2-1)$ matrix A_z whose rows in order are

$$[Y_1, z],\ [X_1, z], \ldots, [Y_{n-1}, z],\ [X_{n-1}, z].$$

A good understanding of A_z is the key to all of our results. We will label the columns of A_z by the identification of z as a row vector in $\mathbb{R}^{n^2-1}$. For $1 \leq j, k \leq n-1$, $[X_j, z]_{kk} = 2iy_{jn}\delta_{jk}$ and $[Y_j, z]_{kk} = -2ix_{jn}\delta_{jk}$. We label the rows $[X_j, z]$ and $[Y_j, z]$ by y_{jn} and x_{jn}, respectively. For $a = x_{jn}$ or y_{jn} and $b = y_{jj}, x_{jk}$ or y_{jk} we denote by (a, b) the entry of A_z along row a and column b. The row and column vectors will be denoted R_a and C_b. We refer to the variables in the set

$$\{y_{ii}, x_{jk}, y_{jk} : 1 \leq i, j, k \leq n-1,\ j < k\}$$

as the nontangent variables, and

$$\{x_{jn}, y_{jn} : 1 \leq j \leq n-1\}$$

as the tangent variables. The submatrix of A_z consisting of the nontangent variable columns will be denoted C_z.

The following lemma describes the elements of A_z in terms of (a,b), along columns given by nontangent variables. The entries in the tangent variable columns could be described as well, but this is not necessary for our main result.

LEMMA 2.3: *Let $z \in \mathbb{R}^{n^2-1}$. For $1 \le j \le n-1$ and $1 \le i < k \le n-1$, we have:*

(i) $(x_{jn}, x_{ik}) = (y_{jn}, x_{ik}) = (x_{jn}, y_{ik}) = (y_{jn}, y_{ik}) = 0$ *if* $i, k \neq j$;

(ii) $(x_{jn}, y_{kk}) = -2x_{jn}\delta_{jk}$, $(y_{jn}, y_{kk}) = 2y_{jn}\delta_{jk}$;

(iii) $(x_{jn}, x_{jk}) = -y_{kn}$, $(x_{jn}, y_{jk}) = -x_{kn}$, $(y_{jn}, x_{jk}) = -x_{kn}$, $(y_{jn}, y_{jk}) = y_{kn}$, *for* $j < k$;

(iv) $(x_{jn}, x_{ij}) = y_{in}$, $(x_{jn}, y_{ij}) = -x_{in}$, $(y_{jn}, x_{ij}) = x_{in}$, $(y_{jn}, y_{ij}) = y_{in}$, *for* $i < j$.

Proof: These results are due to the following calculations:

$$(e_{jn}z)_{mk} = \begin{cases} 0 & \text{if } m \neq j \\ -x_{kn} + iy_{kn} & \text{if } m = j,\ k \neq n \\ iy_{nn} & \text{if } m = j,\ k = n \end{cases};$$

$$(e_{nj}z)_{mk} = \begin{cases} 0 & \text{if } m \neq n \\ -x_{jk} + iy_{jk} & \text{if } m = n,\ k < j \\ iy_{jj} & \text{if } m = n,\ k = j \\ x_{jk} + iy_{jk} & \text{if } m = n,\ k > j \end{cases};$$

$$(ze_{jn})_{mk} = \begin{cases} 0 & \text{if } k \neq n \\ x_{mj} + iy_{mj} & \text{if } k = n,\ m < j \\ iy_{jj} & \text{if } k = n,\ m = j \\ -x_{jm} + iy_{jm} & \text{if } k = n,\ m > j \end{cases};$$

$$(ze_{nj})_{mk} = \begin{cases} 0 & \text{if } k \neq j \\ x_{mn} + iy_{mn} & \text{if } k = j,\ m \neq n \\ iy_{nn} & \text{if } k = j,\ m = n \end{cases}. \qquad \blacksquare$$

We also extend the definition of C_z to any $z \in \mathbb{R}^{n^2-1}$ (where the components of z are as labelled in (2.1)) in the natural way. The matrix C_z for SU(5) is listed in the appendix for the reader's convenience.

LEMMA 2.4: *For any $z \in \mathbb{R}^{n^2-1}$, $(y_{1n}, x_{1n}, \ldots, x_{n-1,n})C_z = 0$.*

Proof: This is equivalent to checking that if R'_a denotes row a of C_z, then

$$\sum_{j=1}^{n-1} y_{jn} R'_{x_{jn}} + \sum_{j=1}^{n-1} x_{jn} R'_{y_{jn}} = 0. \tag{2.2}$$

Expanding the left side of equation (2.2) along the columns labelled by y_{kk}, $k=1,\dots,n-1$, we obtain

$$\sum_{j=1}^{n-1} y_{jn}(-2x_{jn})\delta_{jk} + \sum_{j=1}^{n-1} x_{jn}(2y_{jn})\delta_{jk} = 0.$$

Expanding along the columns x_{ik}, for $1 \le i < k \le n-1$, we have

$$\begin{aligned}&\sum_{j=1}^{n-1} y_{jn}(x_{jn}, x_{ik}) + \sum_{j=1}^{n-1} x_{jn}(y_{jn}, x_{ik})\\ =&y_{in}(x_{in}, x_{ik}) + y_{kn}(x_{kn}, x_{ik}) + x_{in}(y_{in}, x_{ik}) + x_{kn}(y_{kn}, x_{ik})\\ =&y_{in}(-y_{kn}) + y_{kn}(y_{in}) + x_{in}(-x_{kn}) + x_{kn}(x_{in}) = 0.\end{aligned}$$

A similar calculation shows that the expansion along columns y_{ik} is also zero, establishing (2.2). ∎

PROPOSITION 2.5: *Let $z \in O$ be generic.*

(i) *If $z = H$, then $T_H = \operatorname{span}\{e_{jn}-e_{nj}, i(e_{jn}+e_{nj}) : j \le n-1\}$ and $\operatorname{rank}(A_z) = 2(n-1)$.*
(ii) *If $z \neq H$, then $\operatorname{rank}(C_z) = 2n-3$.*

Proof: (i) It is easily seen that for $1 \le j \le n-1$, $[X_j, H] = -ni(e_{jn}+e_{nj})$ and $[Y_j, H] = n(e_{jn}-e_{nj})$. These are linearly independent, hence $\operatorname{rank}(A_H) = 2(n-1)$ and

$$\operatorname{span}\{[X_j,H],\ [Y_j,H] : j \le n-1\} = T_H.$$

(ii) Suppose that for all $1 \le i \le n-1$, $y_{ii} = 1$. Proposition 2.2 shows that in this case $x_{kn} = y_{kn} = 0$ for all $k \neq n$. But then $x_{jk} = y_{jk} = 0$ for all $j \neq k$. As $\operatorname{Tr} z = 0$, this implies $z = H$. Thus for any generic z other than H we may assume that for some i,

$$x_{in}^2 + y_{in}^2 = (1-y_{ii})(1-y_{nn}) \neq 0,$$

and hence at least one of x_{in} or $y_{in} \neq 0$. It follows from the previous lemma that the rows of C_z are linearly dependent, and hence $\operatorname{rank}(C_z) \le 2n-3$.

Consider the following submatrix D_z of C_z:

$$D_z = [C_{y_{ii}}, C_{x_{1i}}, C_{y_{1i}}, \dots, C_{x_{i-1,i}}, C_{y_{i-1,i}}, C_{x_{i,i+1}}, C_{y_{i,i+1}}, \dots, C_{x_{i,n-1}}, C_{y_{i,n-1}}].$$

Note that the number of columns in D_z is $2n-3$. We will show that they are linearly independent. So suppose that

$$\sum_{j=1}^{i-1}\left(a_{ji}C_{x_{ji}} + b_{ji}C_{y_{ji}}\right) + \sum_{j=i+1}^{n-1}\left(a_{ij}C_{x_{ij}} + b_{ij}C_{y_{ij}}\right) + b_{ii}C_{y_{ii}} = 0. \tag{2.3}$$

Expanding (2.3) along rows labelled by x_{kn} for $k \neq i$ we obtain

$$\sum_{j=1}^{i-1} (a_{ji}(x_{kn}, x_{ji}) + b_{ji}(x_{kn}, y_{ji}))$$

$$+ \sum_{j=i+1}^{n-1} (a_{ij}(x_{kn}, x_{ij}) + b_{ij}(x_{kn}, y_{ij})) + b_{ii}(x_{kn}, y_{ii}) = 0. \tag{2.4}$$

Take $1 \leq k < i$. Using Lemma 2.3, (2.4) simplifies to

$$a_{ki}(-y_{in}) + b_{ki}(-x_{in}) = 0.$$

Expanding (2.3) along the row labelled by y_{kn}, $k \neq i$, we find that

$$a_{ki}(-x_{in}) + b_{ki}(y_{in}) = 0.$$

As $x_{in}^2 + y_{in}^2 \neq 0$, we conclude that $a_{ki} = b_{ki} = 0$.

A similar analysis shows that $a_{ik} = b_{ik} = 0$ for $i < k \leq n-1$, hence (2.3) reduces to $b_{ii}C_{y_{ii}} = 0$. But $C_{y_{ii}}(x_{in}) = -2x_{in}$ and $C_{y_{ii}}(y_{in}) = 2y_{in}$. As at least one of x_{in} or $y_{in} \neq 0$, $b_{ii} = 0$. Thus we also have $\operatorname{rank}(C_z) \geq 2n-3$. ∎

Remark 2.1: Although it was not necessary for the proof of this proposition to use the fact that (2.2) holds for all $z \in \mathbb{R}^{n^2-1}$, this will be convenient later in the paper.

We can now find a (small) spanning set for the tangent spaces at generic points. For $z \in O$, let $z_t = [\vec{0}, x_{1n}, y_{1n}, \ldots, x_{n-1,n}, y_{n-1,n}] \in \mathbb{R}^{n^2-1}$.

COROLLARY 2.6: *Suppose $z \in O$ is generic. Then*

$$T_z = \operatorname{span}\{R_{x_{jn}}, R_{y_{jn}}, z_t : j = 1, \ldots, n-1\}.$$

Proof: Note that

$$z_t = \frac{1}{ni}[H, z],$$

and thus belongs to T_z. As remarked above, if $z = H$ the vectors $R_{x_{jn}}, R_{y_{jn}}$, $j = 1, \ldots, n-1$, already span T_z.

Adjoin to A_z the row vector z_t, and let D_z' denote the matrix obtained from this enlarged matrix in the same manner as D_z was obtained from A_z. We already know from the proof of the previous proposition that the columns of D_z' are linearly independent. If $z \neq H$, then (as also noted previously) for some $i \neq n$ at least one of x_{in} or $y_{in} \neq 0$. Hence at least one of $C'_{x_{in}}$ or $C'_{y_{in}}$ is not in the span of the columns of D_z' and therefore $\operatorname{rank}(A_z') \geq 2(n-1)$. ∎

3. Sums of tangent spaces

3.1. KEY RESULTS. In this section we will prove that the $(n-1)$-fold sum of the orbit has measure zero. We will continue to use the notation introduced in section 2. Our main technical result is:

THEOREM 3.1: *If $z^{(1)}, \dots, z^{(n-2)}$ are generic points belonging to the adjoint orbit O of H, then*

$$\dim(T_{z^{(1)}} + \cdots + T_{z^{(n-2)}} + T_H) \leq n^2 - 2.$$

Proof: One can see from the proof of Proposition 2.5(i) that $A_H = [0|\ D_H]$, where $D_H = (d_{jk})$ is a $2(n-1) \times 2(n-1)$ diagonal matrix with $d_{jj} = -ni$ for j odd and $d_{jj} = n$ for j even. It follows from this observation and Corollary 2.6 that the theorem would be proved if we could show that the rank of the $2(n-1)(n-2) \times (n-1)^2$ matrix

$$C(z) \equiv \begin{bmatrix} C_{z^{(1)}} \\ C_{z^{(2)}} \\ \vdots \\ C_{z^{(n-2)}} \end{bmatrix}$$

is at most $n(n-2)$ for all $z = (z^{(1)}, \dots, z^{(n-2)})$, $z^{(j)}$ generic points in O. This is what we now proceed to show.

For any $z^{(j)} \in \mathbb{R}^{n^2-1}$, $1 \leq j \leq n-2$, and $z = (z^{(1)}, \dots, z^{(n-2)})$ define

$$a_j = a_j^{(z)} \equiv \begin{bmatrix} y_{1n}^{(j)} \\ x_{1n}^{(j)} \\ \vdots \\ \vdots \\ y_{n-1,n}^{(j)} \\ x_{n-1,n}^{(j)} \end{bmatrix}, \quad b_j = b_j^{(z)} \equiv \begin{bmatrix} -x_{1n}^{(j)} \\ y_{1n}^{(j)} \\ \vdots \\ \vdots \\ -x_{n-1,n}^{(j)} \\ y_{n-1,n}^{(j)} \end{bmatrix}.$$

Next, we define block vectors, with $(n-2)$ blocks. Each block vector will have either one or two non-zero blocks. We will denote by α_j the block vector whose j-th block is a_j. For $1 \leq i < j \leq n-2$, we set

$$\alpha_{ij} = \alpha_{ij}^{(z)} \equiv \begin{bmatrix} 0 \\ a_j \\ 0 \\ a_i \\ 0 \end{bmatrix}, \quad \beta_{ij} = \beta_{ij}^{(z)} \equiv \begin{bmatrix} 0 \\ b_j \\ 0 \\ -b_i \\ 0 \end{bmatrix}$$

where a_j (resp. b_j) is in block i and a_i ($-b_i$) is in block j. Let

$$K(z)=\{\alpha_i^{(z)},\alpha_{jk}^{(z)},\beta_{jk}^{(z)}:1\leq i,j,k\leq n-2,\ j<k\},$$

and let $Q(z)$ be the transpose of $C(z)$.

There are three main steps to the proof. First, we will show that $K(z)\subseteq\ker(Q(z))$ for all z. Then we will prove that for a dense set of vectors in $\mathbb{R}^{(n^2-1)(n-2)}$, the set $K(z)$ is linearly independent. Lastly, we use these facts to verify that $\operatorname{rank} C(z)\leq n(n-2)$ for all $z=(z^{(1)},\dots,z^{(n-2)})$, $z^{(j)}$ generic points in O.

STEP 1: $K(z)\subseteq\ker(Q(z))$: By symmetry it is enough to show this for $\alpha_1,\alpha_{12},\beta_{12}$. That $\alpha_1\in\ker(Q(z))$ follows from Lemma 2.4.

To prove $\alpha_{12},\beta_{12}\in\ker(Q(z))$ it is enough to show that if

$$C^{(2)}=C^{(2)}(z)\equiv\begin{bmatrix}C_{z^{(1)}}\\C_{z^{(2)}}\end{bmatrix},$$

$Q^{(2)}=\operatorname{transpose}(C^{(2)})$ and

$$\alpha'_{12}\equiv\begin{bmatrix}a_2\\a_1\end{bmatrix},\quad\beta'_{12}\equiv\begin{bmatrix}b_2\\-b_1\end{bmatrix},$$

then $\alpha'_{12},\beta'_{12}\in\ker(Q^{(2)})$.

We will check that $\alpha'_{12}\in\ker(Q^{(2)})$ and leave the verification of β'_{12} to the reader. It suffices to show that $(\alpha'_{12})^{tr}C^{(2)}=0$. For $b=y_{jj},x_{jk},y_{jk}$ we will let $C_b^{(2)}$ denote the column of $C^{(2)}$ labelled by b. Since column $C_{y_{kk}}^{(2)}$ has non-zero entries only in positions $x_{kn}^{(1)},y_{kn}^{(1)},x_{kn}^{(2)},y_{kn}^{(2)}$ with values $-2x_{kn}^{(1)},2y_{kn}^{(1)},-2x_{kn}^{(2)},2y_{kn}^{(2)}$, respectively, it is clear that $(\alpha'_{12})^{\mathrm{tr}}C_{y_{kk}}^{(2)}=0$.

For $1\leq i<k\leq n-1$,

$$\begin{aligned}&(\alpha'_{12})^{\mathrm{tr}}C_{x_{ik}}^{(2)}\\&=\sum_{j=1}^{n-1}\Big(y_{jn}^{(2)}(x_{jn}^{(1)},x_{ik}^{(1)})+x_{jn}^{(2)}(y_{jn}^{(1)},x_{ik}^{(1)})+y_{jn}^{(1)}(x_{jn}^{(2)},x_{ik}^{(2)})+x_{jn}^{(1)}(y_{jn}^{(2)},x_{ik}^{(2)})\Big)\\&=\sum*y_{in}^{(l)}(x_{in}^{(m)},x_{ik}^{(m)})+y_{kn}^{(l)}(x_{kn}^{(m)},x_{ik}^{(m)})+x_{in}^{(l)}(y_{in}^{(m)},x_{ik}^{(m)})+x_{kn}^{(l)}(y_{kn}^{(m)},x_{ik}^{(m)})\end{aligned}$$

where $\sum*$ denotes the sum over $l=1,m=2$ and $l=2,m=1$. Thus

$$(\alpha'_{12})^{\mathrm{tr}}C_{x_{ik}}^{(2)}=\sum*y_{in}^{(l)}(-y_{kn}^{(m)})+y_{kn}^{(l)}(y_{in}^{(m)})+x_{in}^{(l)}(-x_{kn}^{(m)})+x_{kn}^{(l)}(x_{in}^{(m)})=0.$$

The argument that $(\alpha'_{12})^{tr}C_{y_{ik}}^{(2)}=0$ is similar.

STEP 2: The fact that $K(z)$ is linearly independent for "most" z is an immediate consequence of the next two lemmas.

LEMMA 3.2: *Assume that $\{a_1^{(z)}, b_1^{(z)}, \ldots, b_{n-3}^{(z)}, a_{n-2}^{(z)}\}$ is linearly independent. Then $K(z)$ is linearly independent.*

Proof: Assume

$$\sum_{i=1}^{n-2} t_i \alpha_i + \sum_{1 \le i < j \le n-2} t_{ij} \alpha_{ij} + \sum_{1 \le i < j \le n-2} s_{ij} \beta_{ij} = 0. \tag{3.1}$$

By considering the last block one sees that this equality implies

$$t_{n-2} a_{n-2} + \sum_{i=1}^{n-3} (t_{i,n-2} a_i - s_{i,n-2} b_i) = 0,$$

and consequently $t_{n-2} = t_{i,n-2} = s_{i,n-2} = 0$ for all $i = 1, \ldots, n-3$. Substituting this information into (3.1) and examining the 2nd last block yields

$$t_{n-3} a_{n-3} + \sum_{i=1}^{n-4} (t_{i,n-3} a_i - s_{i,n-3} b_i) = 0.$$

Again we conclude that $t_{n-3} = t_{i,n-3} = s_{i,n-3} = 0$ for all $i = 1, \ldots, n-4$. Repeating, it follows that all coefficients $t_i, t_{ij}, s_{ij} = 0$. ∎

LEMMA 3.3: *There is a dense set of $z = (z^{(1)}, \ldots, z^{(n-2)})$, $z^{(j)} \in \mathbb{R}^{n^2-1}$ such that $\{a_1^{(z)}, b_1^{(z)}, \ldots, b_{n-3}^{(z)}, a_{n-2}^{(z)}\}$ is linearly independent.*

Proof: We will assume inductively that for any $x = (x^{(1)}, \ldots, x^{(n-2)})$, $x^{(j)} \in \mathbb{R}^{n^2-1}$, and $\varepsilon > 0$, there is some $z \in \mathbb{R}^{(n^2-1)(n-2)}$ such that $\|z - x\| < \varepsilon$ and the first $2j-1$ rows of the matrix

$$[a_1^{(z)}, -b_1^{(z)}, \ldots, a_j^{(z)}]$$

are linearly independent for all $j \le k$ $(k \le n-2)$. (Of course, then the $2j-1$ column vectors are also linearly independent.)

If $k = 1$, this is easily seen to be true as we only need find z in an ε-neighbourhood of x such that $y_{1n}^{(1)} \ne 0$. Assuming the induction assumption holds for k, we will show that it holds for $k+1$ $(\le n-2)$.

Begin by choosing z in the ε-neighbourhood of x which works for k and consider rows $1, \ldots, 2k-1$ and row $2k+1$ of the matrix

$$[a_1^{(z)}, -b_1^{(z)}, \ldots, -b_k^{(z)}].$$

Notice that the $(2k+1,2k)$ entry of this matrix is $x^{(k)}_{k+1,n}$ and this is formally different from any other entry. By assumption, the upper $(2k-1)\times(2k-1)$ submatrix is invertible, hence there is only one choice for the $(2k+1,2k)$ entry (with all other entries fixed) such that rows $1,\ldots,2k-1,2k+1$ are not linearly independent. If $x^{(k)}_{k+1,n}$ is equal to this one choice, we replace it by a suitable real number such that the new (but still named) vector z continues to satisfy $\|z-x\|<\varepsilon$. As the entries of the upper $(2k-1)\times(2k-1)$ submatrix are unchanged after this replacement, the first $2k-1$ rows of $[a_1^{(z)},-b_1^{(z)},\ldots,a_k^{(z)}]$ remain linearly independent.

Next, we carry out a similar argument with the first $2k+1$ rows of

$$[a_1^{(z)},-b_1^{(z)},\ldots,-b_k^{(z)},\ a_{k+1}^{(z)}].$$

The $(2k,2k+1)$ entry is $x^{(k+1)}_{k,n}$ and this is formally different from any other. The linear independence of rows $1,\ldots,2k-1,\ 2k+1$ of $[a_1^{(z)},-b_1^{(z)},\ldots,b_k^{(z)}]$ again ensures there is only one value for $x^{k+1}_{k,n}$ such that the first $2k+1$ rows of

$$[a_1^{(z)},-b_1^{(z)},\ldots,-b_k^{(z)},\ a_{k+1}^{(z)}]$$

are linearly dependent. Modify this component of z if necessary. Rows $1,\ldots,$ $2k-1$ of $[a_1^{(z)},-b_1^{(z)},\ldots,a_k^{(z)}]$ are unchanged with this new definition of z and thus remain linearly independent. This completes the induction step. ∎

Completion of the Proof of the Theorem.

STEP 3: Bound for $\mathrm{rank}(C(z))$: Steps 1 and 2 clearly imply that for a dense set of $z\in\mathbb{R}^{(n^2-1)(n-2)}$, $K(z)$ is a linearly independent set in $\ker Q(z)$. Thus for a dense set of z,

$$\mathrm{rank}(C(z))=\mathrm{rank}(Q(z))\le 2(n-1)(n-2)-|K(z)|=n(n-2).$$

Suppose there was some vector $x=(x^{(1)},\ldots,x^{(n-2)})$, $x^{(j)}\in\mathbb{R}^{n^2-1}$ with $\mathrm{rank}(C^x)$ $>n(n-2)$. A continuity argument would imply that $\mathrm{rank}(C(z))>n(n-2)$ for all z in a neighbourhood of x, and this is a contradiction. ∎

In order to derive similar results for non-generic points, we need one additional result on the denseness of generic points.

LEMMA 3.4: *The set*

$$\left\{(gHg^{-1},gz^{(1)}g^{-1},\ldots,gz^{(n-2)}g^{-1}):g\in\mathrm{SU}(n),\ z^{(j)}\in O\text{ generic}\right\}$$

is dense in the Cartesian product O^{n-1}.

Proof: Let $x = (gHg^{-1}, gy^{(1)}g^{-1}, \ldots, gy^{(n-2)}g^{-1}) \in O^{n-1}$ where $g \in \mathrm{SU}(n)$, and assume $y^{(j)} = g_j H g_j^{-1}$. Observe that $y^{(j)}$ is generic if and only if $(g_j)_{nn} \neq 0$ (see the proof of Proposition 2.2). In the generic case set $g_j' = g_j$ and $z^{(j)} = y^{(j)}$.

Otherwise, express g_j as $P_j D_j P_j^{-1}$, where $D_j \in \mathrm{SU}(n)$ is diagonal and $P_j \in \mathrm{SU}(n)$. Choose $k \equiv k(j)$ such that $(P_j)_{nk} \neq 0$ and take $i \equiv i(j) \neq k(j)$. Find a neighbourhood Λ_j of 0 such that for all

$$\lambda \in \{e^{i\theta} : \theta \in \Lambda_j,\ \theta \neq 0\} \equiv U_j,$$

we have

$$|(P_j)_{nk}|^2 (D_j)_{kk}\lambda + |(P_j)_{ni}|^2 (D_j)_{ii}\overline{\lambda} \neq |(P_j)_{nk}|^2 (D_j)_{kk} + |(P_j)_{ni}|^2 (D_j)_{ii}.$$

Select $\lambda \in \bigcap_{j=1}^{n-2} U_j$. For g_j non-generic define a diagonal matrix D_j' by

$$(D_j')_{ll} = \begin{cases} (D_j)_{kk}\lambda & \text{if } l = k(j) \\ (D_j)_{ii}\overline{\lambda} & \text{if } l = i(j) \\ (D_j)_{ll} & \text{else} \end{cases}$$

and put $g_j' = P_j D_j' P_j^{-1}$. Then $g_j' \in \mathrm{SU}(n)$ and, as

$$(g_j')_{nn} = \sum_k |(P_j)_{nk}|^2 (D_j')_{kk} \neq (g_j)_{nn},$$

$z^{(j)} = g_j' H g_j'^{-1}$ is generic.

By choosing λ suitably close to 1 we can make

$$(gHg^{-1}, gz^{(1)}g^{-1}, \ldots, gz^{(n-2)}g^{-1})$$

as close to x as necessary. ∎

COROLLARY 3.5: *For all* $z^{(1)}, \ldots, z^{(n-1)} \in O$,

$$\dim(T_{z^{(1)}} + \cdots + T_{z^{(n-1)}}) \leq n^2 - 2 < \dim \underline{\underline{g}}.$$

Proof: Since $T_{\mathrm{Ad(g)z}} = \mathrm{Ad(g)T_z}$, it follows that $\dim \sum_i T_{z^{(i)}} = \dim \sum_i T_{g^{-1}z^{(i)}g}$ for all $g \in G$. Thus $\dim \sum_{i=1}^{n-1} T_{g^{-1}z^{(i)}g} \leq n^2 - 2$ whenever $z^{(1)}, \ldots, z^{(n-2)} \in O$ are generic and $z^{(n-1)} = H$.

Consider the addition map F from O^{n-1} to the $(n-1)$-fold sum of O in $\underline{\underline{g}}$,

$$F\colon O^{n-1} \to (n-1)O \equiv \underbrace{O + \cdots + O}_{n-1} \subseteq \underline{\underline{g}}.$$

The differential of F at $x = (x^{(1)}, \ldots, x^{(n-1)}) \in O^{n-1}$, $(dF)_x$, maps onto the sum of the tangent spaces to O at $x^{(i)}$, $\sum_{i=1}^{n-1} T_{x^{(i)}}$. A continuity argument shows that if the linear map $(dF)_x$ had rank n^2-1 at some x, then it would have rank n^2-1 on a neighbourhood of x. This contradicts the fact that $\dim \sum T_{z^{(i)}} \leq n^2-2$ on a dense set $(z^{(1)}, \ldots, z^{(n-1)}) \in O^{n-1}$. ∎

COROLLARY 3.6: *The $(n-1)$-fold sum of the orbit O has measure zero in $\underline{\underline{g}}$.*

Proof: Again consider the addition map $F\colon O^{n-1} \to (n-1)O \subseteq \underline{\underline{g}}$. Since $\operatorname{rank}(F) \leq n^2-2 < \dim \underline{\underline{g}}$ at all points of O^{n-1}, Sard's theorem ([4]) implies that the measure of the image of F is zero. But the image of F is $(n-1)O$. ∎

3.2. TRANSFERRING THE RESULTS TO SU(n). Finally, we consider the conjugacy class in G containing $\exp(H)$, $C \equiv C(\exp H)$, and the orbital measure $\mu_{\exp H}$ which it supports. Since $\dim C = 2(n-1)$ (see [5], ch. 4: 4.15 or [8], VIII 7.4) we trivially know that C^k has measure 0 if

$$k < \dim G / \dim C = (n+1)/2,$$

and that for such k, $\mu_{\exp H}^k$ is singular to $L^1(G)$. In contrast, as was mentioned in the introduction, $\mu_{\exp H}^n \in L^2$ and hence belongs to L^1 ([3]).

COROLLARY 3.7: (i) *The Haar measure of the C^k is zero if and only if $k \leq n-1$.*
(ii) *The measure $\mu_{\exp H}^k$ is singular to $L^1(G)$ if and only if $k \leq n-1$.*

Proof: By [1], $C^{n-1} \subseteq \exp((n-1)O)$. Since $x \to \exp x$ is a C^1 map and the measure of $(n-1)O$ is zero, it follows that the Haar measure of $\exp((n-1)O)$ is zero. The measure $\mu_{\exp H}^{n-1}$ is singular to L^1, being supported on a set of measure zero.

The measure of C^n is non-zero as C^n supports the non-zero, absolutely continuous measure $\mu_{\exp H}^n$. ∎

As it was already known ([3]) that $\mu^n \in L^1$ for every central, continuous measure μ on SU(n), we have now proven the result stated in the introduction:

COROLLARY 3.8: *When $G = \mathrm{SU}(n)$, $\mu^k \in L^1(G)$ for every central, continuous measure μ on G if and only if $k \geq n$.*

Remark 3.1: Notice that $\mu_{\exp H}^k \in L^1$ if and only if $\mu_{\exp H}^k \in L^2$. It would be interesting to know if this is true for all orbital measures μ_g on SU(n). It is known that if ν is any surface measure supported on a submanifold W such that W^m has interior, then $\nu^m \in L^p$ for some $p > 1$ ([7]).

3.3. N-FOLD SUMS OF TANGENT SPACES. Since $m(C^n) > 0$, it follows from Sard's theorem that

$$\dim(T_{z^{(1)}} + \cdots + T_{z^{(n-1)}} + T_H) = \dim \underline{\underline{g}}$$

for some $z^{(1)}, \ldots, z^{(n-1)} \in O$. In fact, we can display an explicit example of points $z^{(1)}, \ldots, z^{(n-1)}$ for which this is true.

Example 3.1: **Generic points $z^{(1)}, \ldots, z^{(n-1)} \in O$ such that $T_{z^{(1)}} + \cdots + T_{z^{(n-1)}} + T_H = \underline{\underline{g}}$:**

Fix $a, b \in \mathbb{R}\setminus\{0\}$ with $a^2 + b^2 = 1$. For $j = 1, \ldots, n-1$ let $g_j \in \mathrm{SU}(n)$ be defined by

$$(g_j)_{jj} = -a,\ (g_j)_{nn} = a,\ (g_j)_{jn} = b,\ (g_j)_{nj} = b,\ (g_j)_{kk} = 1$$

for $k \neq j, n$, and all other entries of g_j zero. Take

$$z^{(j)} = \mathrm{Ad(g_j)H};$$

we have

$$\begin{aligned} z_{kk}^{(j)} =& i \text{ if } k \neq j, n; \quad z_{jj}^{(j)} = i(1 - nb^2); \quad z_{nn}^{(j)} = i(1 - na^2); \\ z_{jn}^{(j)} =& z_{nj}^{(j)} = -abni; \quad \text{and } z_{ik}^{(j)} = 0 \text{ else.} \end{aligned}$$

Certainly $z^{(j)}$ is generic.

As in section 2 let

$$X_k = e_{kn} - e_{nk} \quad \text{and} \quad Y_k = i(e_{kn} + e_{nk}),$$

and recall that $[X_k, z^{(j)}], [Y_k, z^{(j)}] \in T_{z^{(j)}}$. From Lemma 2.3 one can check that for $k \neq j, n$,

$$\begin{aligned} [X_k, z^{(j)}] =& -iabn(e_{kj} + e_{jk}) + n(a^2 - b^2)(e_{jn} - e_{nj}), \\ [Y_k, z^{(j)}] =& abn(e_{kj} - e_{jk}) + n(a^2 - b^2)(e_{kn} - e_{nk}), \end{aligned}$$

while

$$[X_j, z^{(j)}] = -2iabn(e_{jj} - e_{nn}) + n(a^2 - b^2)(e_{jn} - e_{nj}).$$

We observed earlier in the paper that

$$T_H = \mathrm{span}\{e_{jn} - e_{nj},\ i(e_{jn} + e_{nj}) : 1 \leq j \leq n-1\}.$$

Thus all the vectors $i(e_{ll} - e_{nn})$, $e_{jk} - e_{kj}$ and $i(e_{jk} + e_{kj})$ for $j < k \leq n$, $l < n$ belong to $T_{z^{(1)}} + \cdots + T_{z^{(n-1)}} + T_H$, and these clearly span $\underline{\underline{g}}$.

4. Appendix — The transpose of matrix C_z for SU(5)

	Rx_{15}	Ry_{15}	Rx_{25}	Ry_{25}	Rx_{35}	Ry_{35}	Rx_{45}	Ry_{45}
Cy_{11}	$-2x_{15}$	$2y_{15}$	0	0	0	0	0	0
Cy_{22}	0	0	$-2x_{25}$	$2y_{25}$	0	0	0	0
Cy_{33}	0	0	0	0	$-2x_{35}$	$2y_{35}$	0	0
Cy_{44}	0	0	0	0	0	0	$-2x_{45}$	$2y_{45}$
Cx_{12}	$-y_{25}$	$-x_{25}$	y_{15}	x_{15}	0	0	0	0
Cy_{12}	$-x_{25}$	y_{25}	$-x_{15}$	y_{15}	0	0	0	0
Cx_{13}	$-y_{35}$	$-x_{35}$	0	0	y_{15}	x_{15}	0	0
Cy_{13}	$-x_{35}$	y_{35}	0	0	$-x_{15}$	y_{15}	0	0
Cx_{14}	$-y_{45}$	$-x_{45}$	0	0	0	0	y_{15}	x_{15}
Cy_{14}	$-x_{45}$	y_{45}	0	0	0	0	$-x_{15}$	y_{15}
Cx_{23}	0	0	$-y_{35}$	$-x_{35}$	y_{25}	x_{25}	0	0
Cy_{23}	0	0	$-x_{35}$	y_{35}	$-x_{25}$	y_{25}	0	0
Cx_{24}	0	0	$-y_{45}$	$-x_{45}$	0	0	y_{25}	x_{25}
Cy_{24}	0	0	$-x_{45}$	y_{45}	0	0	$-x_{25}$	y_{25}
Cx_{34}	0	0	0	0	$-y_{45}$	$-x_{45}$	y_{35}	x_{35}
Cy_{34}	0	0	0	0	$-x_{45}$	y_{45}	$-x_{35}$	y_{35}

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